Elementary Techniques in differentiation from the First Principle

Yue Kwok Choy

1. Grouping

Question $f(x) = 4x^3 - 3x^2$, find $f'(x)$.

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{\left[4(x+\Delta x)^3 - 3(x+\Delta x)^2\right] - \left[4x^3 - 3x^2\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{4\left[(x+\Delta x)^3 - x^3\right] - 3\left[(x+\Delta x)^2 - x^2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{4\left[(x+\Delta x)^3 - x^3\right] - 3\left[(x+\Delta x)^2 - x^2\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{4\Delta x\left[(x+\Delta x)^2 + (x+\Delta x)x + x^2\right] - 3\Delta x\left[(x+\Delta x) + x\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left\{4\left[(x+\Delta x)^2 + (x+\Delta x)x + x^2\right] - 3\left[(x+\Delta x) + x\right]\right\} = 4\left[3x^2\right] - 3\left[2x\right] = \underline{\underline{12x^2 - 6x}}$$

2. Joining fractions

Question $f(x) = \frac{1}{x-1}$, find $f'(x)$.

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x-1} - \frac{1}{x-1}}{\Delta x} = \lim_{\Delta x \to 0} \frac{x-1-(x+\Delta x-1)}{\Delta x(x+\Delta x-1)(x-1)} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x(x+\Delta x-1)(x-1)}$$

$$= -\lim_{\Delta x \to 0} \frac{1}{(x+\Delta x-1)(x-1)} = \underline{\underline{-\frac{1}{(x-1)^2}}}$$

3. Rationalize the numerator

Question $f(x) = \sqrt{2x-1}$, find $f'(13)$.

Solution

$$f'(13) = \lim_{\Delta x \to 0} \frac{\sqrt{2(13+\Delta x)-1} - \sqrt{2(13)-1}}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{25+2\Delta x} - 5}{\Delta x} = \lim_{\Delta x \to 0} \frac{\left[\sqrt{25+2\Delta x} - 5\right]\left[\sqrt{25+2\Delta x} + 5\right]}{\Delta x\left[\sqrt{25+2\Delta x} + 5\right]}$$

$$= \lim_{\Delta x \to 0} \frac{25 + 2\Delta x - 25}{\Delta x\left[\sqrt{25+2\Delta x} + 5\right]} = \lim_{\Delta x \to 0} \frac{2}{\sqrt{25+2\Delta x} + 5} = \frac{2}{5+5} = \underline{\underline{\frac{1}{5}}}$$

Question $f(x) = \sqrt[3]{x} - \frac{1}{x}$, find $f'(x)$.

Solution

$$f'(x) = \lim_{\Delta x \to 0} \frac{\left[\sqrt[3]{x+\Delta x} - \frac{1}{x+\Delta x}\right] - \left[\sqrt[3]{x} - \frac{1}{x}\right]}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt[3]{x+\Delta x} - \sqrt[3]{x}}{\Delta x} + \lim_{\Delta x \to 0} \frac{\frac{1}{x} - \frac{1}{x+\Delta x}}{\Delta x} \qquad \textbf{(Grouping)}$$

$$= \lim_{\Delta x \to 0} \frac{\left[(x+\Delta x)^{1/3} - x^{1/3}\right]\left[(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}\right]}{\Delta x\left[(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}\right]} + \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{(\Delta x)x(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{(x+\Delta x) - x}{\Delta x\left[(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}\right]} + \lim_{\Delta x \to 0} \frac{\Delta x}{(\Delta x)x(x+\Delta x)}$$

$$= \lim_{\Delta x \to 0} \frac{1}{(x+\Delta x)^{2/3} + x^{1/3}(x+\Delta x)^{1/3} + x^{2/3}} + \lim_{\Delta x \to 0} \frac{1}{x(x+\Delta x)} = \underline{\underline{\frac{1}{3x^{2/3}} + \frac{1}{x^2}}}$$

4. Absolute value

Question $f(x) = |x - 2|$, find $f'(x)$.

Solution

Case 1 When $x > 2$, $f(x) = x - 2$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x - 2) - (x - 2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Case 2 When $x < 2$, $f(x) = -(x - 2)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{[-(x + \Delta x - 2)] - [-(x - 2)]}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{\Delta x} = -1$$

Case 3 When $x = 2$,

$$f'_{+}(2) = \lim_{\Delta x \to 0^{+}} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0^{=}} \frac{|\Delta x| - 0}{\Delta x} = \lim_{\Delta x \to 2^{-}} \frac{\Delta x}{\Delta x} = 1$$

$$f'_{-}(2) = \lim_{\Delta x \to 0^{-}} \frac{f(2+\Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0^{-}} \frac{|\Delta x| - 0}{\Delta x} = \lim_{\Delta x \to 2^{-}} \frac{-\Delta x}{\Delta x} = -1$$

Since the right-handed and left-derivatives are unequal, therefore f(x) is not differentiable at $x = 0$.

$$\therefore \quad f'(x) = \begin{cases} 1 & ,\text{when } x > 2 \\ -1 & ,\text{when } x < 2 \\ \text{undefined} & ,\text{when } x = 2 \end{cases} .$$

5. **Trigonometry**

Question f(x) = cos x , find f '(x).

Solution

$$f'(x) = \lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x} = \lim_{\Delta x\to 0}\frac{-2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\Delta x} = -\lim_{\Delta x\to 0}\sin\left(x+\frac{\Delta x}{2}\right)\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}}$$

$$= (-\sin x)(1) = \underline{\underline{-\sin x}}$$

Question f(x) = tan x , find $f'\left(\frac{\pi}{4}\right)$.

Solution

Method 1

$$f'(x) = \lim_{\Delta x\to 0}\frac{\tan(x+\Delta x)-\tan x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{\tan x+\tan\Delta x}{1+\tan x\tan\Delta x}-\tan x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\tan\Delta x(1+\tan^2 x)}{\Delta x(1-\tan x\tan\Delta x)}$$

$$= \lim_{\Delta x\to 0}\frac{\tan\Delta x}{\Delta x}\lim_{\Delta x\to 0}\frac{\sec^2 x}{1-\tan x\tan\Delta x} = 1\times\sec^2 x = \sec^2 x$$

$$f'\left(\frac{\pi}{4}\right) = \sec^2\left(\frac{\pi}{4}\right) = \underline{\underline{2}}$$

Method 2

$$f'\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}}\frac{\tan x-\tan\frac{\pi}{4}}{x-\frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}}\frac{\tan\left(x-\frac{\pi}{4}\right)\left(1+\tan x\tan\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}$$

$$= \lim_{x\to\frac{\pi}{4}}\frac{\sin\left(x-\frac{\pi}{4}\right)}{x-\frac{\pi}{4}}\lim_{x\to\frac{\pi}{4}}\frac{1}{\cos\left(x-\frac{\pi}{4}\right)}\lim_{x\to\frac{\pi}{4}}\left(1+\tan x\tan\frac{\pi}{4}\right) = 1\times\frac{1}{1}\times(1+1) = \underline{\underline{2}}$$

Question f(x) = sec x , find f '(x) ..

Solution

$$f'(x) = \lim_{\Delta x\to 0}\frac{\sec(x+\Delta x)-\sec x}{\Delta x} = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{1}{\cos(x+\Delta x)}-\frac{1}{\cos x}\right] = \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{\cos x-\cos(x+\Delta x)}{\cos(x+\Delta x)\cos x}\right]$$

$$= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left[\frac{2\sin\left(x+\frac{\Delta x}{2}\right)\sin\frac{\Delta x}{2}}{\cos(x+\Delta x)\cos x}\right] = \lim_{\Delta x\to 0}\frac{\sin\left(x+\frac{\Delta x}{2}\right)}{\cos(x+\Delta x)\cos x}\lim_{\Delta x\to 0}\frac{\sin\frac{\Delta x}{2}}{\frac{\Delta x}{2}} = \frac{\sin x}{\cos^2 x} = \underline{\underline{\sec x\tan x}}$$